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Approximation of linear conflict-controlled neutral-type systems. ⋆

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Abstract: We consider a dynamical system described by linear neutral-type functional-differential equations which is controlled under conditions of unknown disturbances. This system is approximated by a system of ordinary differential equations. An aiming procedure between the initial and approximating systems is elaborated. Using such procedure, results of the control theory for ordinary differential systems can be applied to control of the initial system.

Keywords: approximation scheme, feedback control, neutral-type equations, differential games.

1. INTRODUCTION

The research of approximations of delay differential equations by ordinary differential equations have an extensive history. The convergence of such approximations for linear systems with constant delays was proved in Krasovskii (1964). In Repin (1965), this result was extended to nonlinear systems, and in Kurjanskii (1967) — to the case of variable delays. Later, similar approximations, their generalizations and applications to different kinds of problems were considered, for example, by Kryajimskii and Maximov (1978); Banks and Burns (1978); Banks and Kappel (1979); Kunisch (1980); Fabiano (2013). Note that, approximations of neutral-type functional differential equations were considered in Kunisch (1980); Fabiano (2013). It was proposed in Krasovskii and Kotelnikova (2011) to use the approximating system of ordinary differential equations as a leader for the initial time-delay dynamical system controlled under conditions of disturbances or counteractions. In this case, an auxiliary problem arises of the aiming between the motion of the initial conflict-controlled system and the motion of the approximating system. A solution of this problem was given for systems described by different functional differential equations in Lukoyanov and Plaksin (2015); Plaksin (2015); Lukoyanov and Plaksin (2016). This present paper continues these investigations and is devoted to the solution of this problem for dynamical systems described by linear neutral-type functional differential equations in a quite general form.

2. NOTATIONS

By $\mathbb{R}^n$ we denote Euclidean space of n-dimensional vectors; $\mathbb{R}^{m\times n}$ is the linear space of $m\times n$-matrices; double vertical bars $\|\cdot\|$ denote the Euclidean norm of vectors from $\mathbb{R}^n$ and the consistent norm of matrices from $\mathbb{R}^{m\times n}$. The angle brackets $\langle\cdot,\cdot\rangle$ denote the inner product of vectors. By $AC([a,b],\mathbb{R}^n)$ and $L([a,b],\mathbb{R}^n)$ we denote, respectively, the sets of absolutely continuous and Lebesgue integrable functions from $[a,b]$ to $\mathbb{R}^n$.

3. CONFLICT-CONTROLLED SYSTEM

Consider a conflict-controlled dynamical system described by the linear neutral-type functional-differential equation

$$\dot{x}[t] - \mathbf{L}_1(t,\dot{x}_t[\cdot]) = \mathbf{L}_2(t,x_t[\cdot]) + P[t]u[t] + Q[t]v[t] + g[t],$$
$$t\in[t_0,\vartheta],\quad x[t]\in\mathbb{R}^n,\quad u[t]\in\mathbb{U},\quad v[t]\in\mathbb{V}, \tag{1}$$

with the initial condition

$$x[t_0]=\theta,\ \ x[t_0+\xi]=\phi[\xi],\ \ \dot{x}[t_0+\xi]=\psi[\xi],\ \ \xi\in[-h,0),$$
$$(\theta,\phi[\cdot],\psi[\cdot])\in\Theta. \tag{2}$$

Here t is the time variable; t_0, ϑ are fixed instants of time; $x[t]$ is the value of the state vector at the moment t; $\dot{x}[t]=\mathrm{d}x[t]/\mathrm{d}t$ for $t\in[t_0,\vartheta]$; $h>0$ is the delay constant; $x_t[\cdot]$ is the motion history on the interval $[t-h,t]$ defined by $x_t[\xi]=x[t+\xi]$, $\xi\in[-h,0]$; analogously, $\dot{x}_t[\cdot]$ is the derivative history on $[t-h,t]$; $u[t]$ and $v[t]$ are respectively the current control and disturbance actions; $\mathbb{U}\subset\mathbb{R}^{n_u}$, $\mathbb{V}\subset\mathbb{R}^{n_v}$ are known compact sets, where $n_u,n_v\in\mathbb{N}$; The operators $\mathbf{L}_i\colon[t_0,\vartheta]\times L([-h,0],\mathbb{R}^n)\mapsto\mathbb{R}^n$, $i=1,2$, are defined by the following formula:

$$\mathbf{L}_i(t,w[\cdot]) = \sum_{j=0}^{k} A_{i,j}[t]w[-h_j] + \int_{-h}^{0} B_i[t,\xi]w[\xi]\,\mathrm{d}\xi, \tag{3}$$

where $0=h_0<h_1<\ldots<h_k=h$; for each $i=1,2$, $j=\overline{0,k}$ matrix-functions the $A_{i,j}[\cdot]\colon[t_0,\vartheta]\mapsto\mathbb{R}^{n\times n}$ are measurable, and there exist $\alpha_{i,j}>0$ such that

$$\|A_{i,j}[t]\|\le\alpha_{i,j} \text{ for a.e. } t\in[t_0,\vartheta]; \tag{4}$$

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for each $i = 1, 2$, the matrix-functions $B_i[\cdot,\cdot]\colon [t_0,\vartheta] \times [-h, 0] \mapsto \mathbb{R}^{n\times n}$ are continuous in the first argument and measurable in the second argument, and there exist $\beta_i > 0$ such that

$$\|B_i[t,\xi]\| \leq \beta_i \text{ for a.e. } \xi \in [-h,0], \quad t \in [t_0,\vartheta]; \tag{5}$$

the matrix-functions $P[\cdot]\colon [t_0,\vartheta] \mapsto \mathbb{R}^{n\times n_u}$ and $Q[\cdot]\colon [t_0,\vartheta] \mapsto \mathbb{R}^{n\times n_v}$ are continuous; the function $g[\cdot]\colon [t_0,\vartheta] \mapsto \mathbb{R}^n$ is measurable and satisfies the inequality

$$\|g[t]\| \leq \gamma \text{ for a.e. } t \in [t_0,\vartheta]; \tag{6}$$

set Θ consists of triples $(\theta,\phi[\cdot],\psi[\cdot]) \in \mathbb{R}^n \times L([-h,0],\mathbb{R}^n) \times L([-h,0],\mathbb{R}^n)$ such that

$$\|\theta\| \leq R_0, \quad \|\phi[\xi]\| \leq R_0, \quad \|\psi[\xi]\| \leq R_0 \quad \text{for a.e. } \xi \in [-h,0). \tag{7}$$

where R_0 is a fixed constant.

Furthermore, the following conditions are assumed:

(**L**.1). There exists a number $h_* \in (0,h_1)$ such that

$$A_{1,0}[t] = 0, \quad B_1[t,\xi] = 0, \quad \xi \in [-h_*,0], \quad t \in [t_0,\vartheta].$$

(**L**.2). For the numbers $\alpha_{1,j}$, $j = \overline{1,k}$ and β_1 from (4), (5), the following inequality holds:

$$\sum_{j=1}^{k} \alpha_{1,j} + h\beta_1 < 1.$$

Measurable functions $u[\cdot]\colon [t_0,\vartheta) \mapsto \mathbb{U}$ and $v[\cdot]\colon [t_0,\vartheta) \mapsto \mathbb{V}$ are called admissible realizations of the control actions $u[t]$ and the disturbance actions $v[t]$, respectively.

For $R > 0$, the following notation will be used:

$$AC(R) = \big\{x[\cdot] \in AC([t_0,\vartheta],\mathbb{R}^n)\colon \|x[t]\| \leq R,\ t \in [t_0,\vartheta], \|\dot{x}[t]\| \leq R \text{ for a.e. } t \in [t_0,\vartheta]\big\}.$$

Under the conditions above, it can be shown (following, for example, the scheme from Filippov (1988), p. 3) that, for any initial data $(\theta,\phi[\cdot],\psi[\cdot]) \in \Theta$ and any admissible realizations $u[\cdot]$ and $v[\cdot]$ there exists a unique solution $x[\cdot]\colon [t_0,\vartheta] \mapsto \mathbb{R}^n$ of problem (1), (2), which is an absolutely continuous function, which satisfies the equality $x[t_0] = \theta$ and almost everywhere satisfies equation (1), in which, according to (2), $x_t[\xi]$ is replaced by $\phi[\xi]$ and $\dot{x}_t[\xi]$ is replaced by $\psi[\xi]$ when $t+\xi < t_0$. Moreover, there exists a number $R_x > 0$ such that any solution $x[\cdot]$ of problem (1), (2) satisfies the relation

$$x[\cdot] \in AC(R_x). \tag{8}$$

4. APPROXIMATING SYSTEM

Let $m \in \mathbb{N}$, $\Delta h = h/m$. For a vector $Y = (y^{[1]},\ldots,y^{[m]}) \in \mathbb{R}^{mn}$, we denote by $S(Y)[\cdot]$ the function that is the linear spline on $[-h,0]$ with nodes $-i\Delta h$, $i = \overline{0,m}$ such that

$$S(Y)[0] = y^{[1]}, \quad S(Y[t])[-i\Delta h] = y^{[i]}, \quad i = \overline{1,m}. \tag{9}$$

Applying the approximation idea from (Lukoyanov and Plaksin, 2015, Section 3) to system (1) we construct the following system of linear equations:

$$\begin{cases} z^{[0]}[t] = \mathbf{L}_1(t, S(Z[t])[\cdot]) + \mathbf{L}_2(t, S(Y[t])[\cdot]) \\ \qquad\qquad + P[t]\hat{u}[t] + Q[t]\hat{v}[t] + g[t], \\ \dot{y}^{[0]}[t] = z^{[0]}[t], \\ \dot{y}^{[i]}[t] = (y^{[i-1]}[t] - y^{[i]}[t])/\Delta h,\ i = \overline{1,m}, \\ \dot{z}^{[i]}[t] = (z^{[i-1]}[t] - z^{[i]}[t])/\Delta h,\ i = \overline{1,m}, \end{cases} \tag{10}$$

$$t \in [t_0,\vartheta],\ y^{[i]}[t], z^{[i]}[t] \in \mathbb{R}^n,\ i = \overline{0,m},\ \hat{u}[t] \in \mathbb{U},\ \hat{v}[t] \in \mathbb{V},$$

$$Y[t] = (y^{[1]}[t],\ldots,y^{[m]}[t]),\ Z[t] = (z^{[1]}[t],\ldots,z^{[m]}[t]).$$

The initial condition for system (10) is determined by a triplet $(\theta,\phi[\cdot],\psi[\cdot]) \in \Theta$ from (7) according to the rule

$$y^{[0]}[t_0] = \theta, \quad y^{[i]}[t_0] = \frac{1}{\Delta h} \int_{-i\Delta h}^{-(i-1)\Delta h} \phi[\xi]\, \mathrm{d}\xi, \tag{11}$$

$$z^{[i]}[t_0] = \frac{1}{\Delta h} \int_{-i\Delta h}^{-(i-1)\Delta h} \psi[\xi]\, \mathrm{d}\xi, \quad i = \overline{1,m}.$$

Due to condition (**L**.1) and definition (9) of the splines $S(Y[t])[\cdot]$ and $S(Z[t])[\cdot]$, system (11) can be reduced to the system of linear ordinary differential equations with the state vector

$$W = (y^{[0]}, y^{[1]},\ldots,y^{[m]}, z^{[1]},\ldots,z^{[m]}) \in \mathbb{R}^{(1+2m)n}.$$

Consequently, for any $(\theta,\phi[\cdot],\psi[\cdot]) \in \Theta$ and any admissible realizations $\hat{u}[\cdot]$ and $\hat{v}[\cdot]$, there exists a unique solution $W[\cdot]\colon [t_0,\vartheta] \mapsto \mathbb{R}^{(1+2m)n}$ of problem (10), (11), which is an absolutely continuous function satisfying initial condition (11) and equations (10) almost everywhere.

Let us define

$$y^{[0]}[t] = \phi[t-t_0], \quad z^{[0]}[t] = \psi[t-t_0], \quad t \in [t_0-h,t_0), \tag{12}$$

Lemma 1. There exists a number $R_y > 0$ such that, for any $m \in \mathbb{N}$, any $(\theta,\phi[\cdot],\psi[\cdot]) \in \Theta$ and any admissible realizations $\hat{u}[\cdot]$ and $\hat{v}[\cdot]$, the following inequalities hold:

$$\|y^{[0]}[t]\| \leq R_y,\ \|z^{[0]}[t]\| \leq R_y \text{ for a.e. } t \in [t_0-h,\vartheta],$$

$$\|y^{[i]}[t]\| \leq R_y, \quad \|z^{[i]}[t]\| \leq R_y, \quad t \in [t_0,\vartheta], \quad i = \overline{1,m}.$$

Proof. This lemma can be proved by the scheme from (Plaksin, 2015, Lemma 3), if we take into account inequalities (4)–(7) and conditions (**L**.1), (**L**.2). ■

Theorem 1. For any numbers $R_r > 0$ and $\varepsilon > 0$, there exists a number $M > 0$ such that, for any $r[\cdot] \in AC(R_r)$, any $m \geq M$, any $(\theta,\phi[\cdot],\psi[\cdot]) \in \Theta$, any admissible realizations $\hat{u}[\cdot]$, $\hat{v}[\cdot]$, and any $t \in [t_0,\vartheta]$, the following inequalities hold:

$$\begin{aligned} \left| \int_{t_0}^{t} \langle \mathbf{L}_1(\tau, z^{[0]}_\tau[\cdot] - S(Z[\tau])[\cdot]), r[\tau]\rangle\, \mathrm{d}\tau \right| \leq \varepsilon, \\ \left| \int_{t_0}^{t} \langle \mathbf{L}_2(\tau, y^{[0]}_\tau[\cdot] - S(Y[\tau])[\cdot]), r[\tau]\rangle\, \mathrm{d}\tau \right| \leq \varepsilon. \end{aligned} \tag{13}$$

Proof. Let us prove the first inequality. By (3), we have

$$\int_{t_0}^{t}\langle \mathbf{L}_1(\tau, z_\tau^{[0]}[\cdot] - S(Z[\tau])[\cdot]), r[\tau]\rangle \,\mathrm{d}\tau \tag{14}$$

$$= \sum_{j=0}^{k}\int_{t_0}^{t}\langle z_\tau^{[0]}[-h_j] - S(Z[\tau])[-h_j], A^*_{1,j}[\tau]r[\tau]\rangle \,\mathrm{d}\tau$$

$$+ \int_{-h}^{0}\int_{t_0}^{t}\langle z_\tau^{[0]}[\xi] - S(Z[\tau])[\xi], B^*_1[\tau,\xi]r[\tau]\rangle \,\mathrm{d}\tau \,\mathrm{d}\xi,$$

where $A^*_{1,j}[\tau]$ and $B^*_1[\tau,\xi]$ are the matrices conjugated to $A_{1,j}[\tau]$ and $B_1[\tau,\xi]$, respectively.

Using Lemma 1 and definition (9) of $S(Z[\tau])[\cdot]$, we have

$$\|z_\tau^{[0]}[\xi] - S(Z[\tau])[\xi]\| \le 2R_y,\ \xi\in[-h,0],\ \tau\in[t_0,\vartheta]. \tag{15}$$

For each $j = \overline{0,k}$ and $\xi\in[-h,0]$, there exist (Natanson, 1960, p. 214) numbers $R_a, R_b > 0$ and functions $a_j[\cdot]\in AC(R_a)$ and $b[\cdot,\xi]\in AC(R_b)$ such that

$$\begin{aligned}&\int_{t_0}^{\vartheta}\|A^*_{1,j}[\tau]r[\tau] - a_j[\tau]\|\,\mathrm{d}\tau \le \frac{\varepsilon}{8R_yR_rk},\\ &\int_{t_0}^{\vartheta}\|B^*_1[\tau,\xi]r[\tau] - b[\tau,\xi]\|\,\mathrm{d}\tau \le \frac{\varepsilon}{8R_yR_rh}.\end{aligned} \tag{16}$$

Let $R_c = \max\{R_a, R_b\}$. Let us define the functions

$$\begin{aligned}c_j[\tau,\xi] = a_j[\tau],\quad j=\overline{0,k},\quad c_{k+1}[\tau,\xi] = b[\tau,\xi],\\ \xi\in[-h,0],\quad \tau\in[t_0,\vartheta].\end{aligned} \tag{17}$$

Then we have $c_j[\cdot,\xi]\in AC(R_c)$, $j=\overline{0,k+1}$, $\xi\in[-h,0]$. From (14), (15)–(17) we deduce

$$\left|\int_{t_0}^{t}\langle \mathbf{L}_1(\tau, z_\tau^{[0]}[\cdot] - S(Z[\tau])[\cdot]), r[\tau]\rangle \,\mathrm{d}\tau\right| \le \varepsilon/2$$

$$+\sum_{j=0}^{k}\left|\int_{t_0}^{t}\langle z_\tau^{[0]}[-h_j] - S(Z[\tau])[-h_j], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|$$

$$+\int_{-h}^{0}\left|\int_{t_0}^{t}\langle z_\tau^{[0]}[\xi] - S(Z[\tau])[\xi], c_{k+1}[\tau,\xi]\rangle \,\mathrm{d}\tau\right|\mathrm{d}\xi.$$

Let us fix $t\in[t_0,\vartheta]$, $\xi\in[-h,0]$ and $j=\overline{0,k+1}$. Let $i=\overline{1,m}$ be such that $\xi\in[-i\Delta h, -(i-1)\Delta h]$. Then, for any $t\in[t_0,\vartheta]$, we have

$$\begin{aligned}&\left|\int_{t_0}^{t}\langle z_\tau^{[0]}[\xi] - S(Z[\tau])[\xi], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|\\ &\le \left|\int_{t_0}^{t}\langle z_\tau^{[0]}[\xi] - z_\tau^{[0]}[-i\Delta h], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|\\ &\quad+\left|\int_{t_0}^{t}\langle z_\tau^{[0]}[-i\Delta h] - z^{[i]}[\tau], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|\\ &\quad+\left|\int_{t_0}^{t}\langle z^{[i]}[\tau] - S(Z[\tau])[\xi], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|.\end{aligned} \tag{18}$$

Let us estimate each term in this formula. Denote $t^i_\xi = (i+1)\Delta h + \xi$, $R_1 = (2+\vartheta-t_0)R_yR_c$. Using Lemma 1 and the inclusion $c_j[\cdot,\xi]\in AC(R_c)$, for the first term, we have

$$\left|\int_{t_0}^{t}\langle z_\tau^{[0]}[\xi] - z_\tau^{[0]}[-(i+1)\Delta h], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|$$

$$= \left|\int_{t_0}^{t}\langle z_\tau^{[0]}[\xi], c_j[\tau,\xi]\rangle\mathrm{d}\tau - \int_{t_0-t^i_\xi}^{t-t^i_\xi}\langle z_\tau^{[0]}[\xi], c_j[\tau+t^i_\xi,\xi]\rangle\mathrm{d}\tau\right| \le R_1\Delta h.$$

There exists (Plaksin, 2015, Theorem 2) a number $M_* = M_*(\varepsilon) > 0$ such that, for any $m \ge M_*$, for the second term, the following inequality is valid:

$$\left|\int_{t_0}^{t}\langle z_\tau^{[0]}[-(i+1)\Delta h] - z^{[i]}[\tau], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right| \le \varepsilon/3.$$

Due to definition (9) of $S(Z[\tau])[\cdot]$, system (11), applying the integration by parts formula and the inclusion $c_i[\cdot,\xi]\in AC(R_c)$, for the third term, we deduce

$$\left|\int_{t_0}^{t}\langle z^{[i]}[\tau] - S(Z[\tau])[\xi], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right|$$

$$\le \Delta h\left|\int_{t_0}^{t}\langle \dot z^{[i]}[\tau], c_j[\tau,\xi]\rangle \,\mathrm{d}\tau\right| \le \Delta h\left|\int_{t_0}^{t}\langle z^{[i]}[\tau], \frac{\mathrm{d}c_j[\tau,\xi]}{\mathrm{d}\tau}\rangle \,\mathrm{d}\tau\right|$$

$$+2\Delta h\max_{\tau\in[t_0,t]}\Big(\|z^{[i]}[\tau]\|\|c_j[\tau,\xi]\|\Big) \le R_1\Delta h.$$

Put $M = \max\{M_*, 4(k+h+1)R_1/\varepsilon\}$. Then, for any $m \ge M$, the first inequality in (13) holds. The second inequality in (13) can be proved in a similar way. ■

5. AUXILIARY LEMMA

Denote $T[t] = \max\{t_0, t\}$, $t\in[t_0-h,\vartheta]$.

Lemma 2. Let numbers $0 < h_* < h_1 < \ldots < h_k$ from (3) and (**L**.1), numbers $R_s > 0$ and $\eta > 0$ be fixed. Let matrix-functions $F_j[\cdot]\colon [t_0,\vartheta]\mapsto\mathbb{R}^{n\times n}$, $j=\overline{1,k}$ be measurable and there exists a number $\alpha_* > 0$ such that

$$\|F_j[t]\| \le \alpha_* \text{ for a.e. } t\in[t_0,\vartheta]; \tag{19}$$

a matrix-function $G[\cdot,\cdot]\colon [t_0,\vartheta]\times[t_0,\vartheta]\mapsto\mathbb{R}^{n\times n}$ be continuous. Then for any number $\varepsilon > 0$, there exists a number $\nu = \nu(\varepsilon) > 0$ such that for any $s[\cdot]$ satisfying the relations

$$s[\cdot]\in AC(R_s),\quad \|s[t]\| \le \nu + \eta\int_{t_0}^{t}\|s[\tau]\|\,\mathrm{d}\tau \tag{20}$$

$$+\sum_{i=1}^{k}\left\|\int_{t_0}^{T[t-h_i]}F_i[\tau]\frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau}\,\mathrm{d}\tau\right\| + \left\|\int_{t_0}^{T[t-h_*]}G[t,\tau]\frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau}\,\mathrm{d}\tau\right\|,$$

the following estimate holds:

$$\|s[t]\| \le \varepsilon,\quad t\in[t_0,\vartheta]. \tag{21}$$

Proof. Let a number $l\in\mathbb{N}$ be such that $(\vartheta - t_0)/h_* < l$. Let us define $F_j[\tau] = F_j[\vartheta]$ and $G[t,\tau] = G[t,\vartheta]$ for $\tau\in(\vartheta, t_0+lh_*]$, $t\in[t_0,\vartheta]$. Denote

$$t_i = t_0 + ih_*,\quad t_{i,j} = T[t_i + (h_* - h_j)],\quad i=\overline{1,l},\quad j=\overline{1,k}.$$

If $l=1$, then, from (20), due to Gronwall-Bellman lemma (Bellman and Cooke, 1963, p. 31), it follows that estimate (21) holds for $\nu = \varepsilon e^{-\eta(\vartheta - t_0)}$.

Let $l \geq 2$. Define a number $\mu_{l-1} = 1$. Successively, for any $i = l-2, l-3, \ldots, 0$, let us define the numbers

$$\varepsilon_i = \varepsilon e^{-\eta(\vartheta - t_0)} / (6R_s(l-1)\mu_{l-1}\mu_{l-2}\ldots\mu_{i+1}), \qquad (22)$$

Lipschitz continuous matrix-functions $\widetilde{F}_{i,j}[\cdot]\colon [t_{i,j}, t_{i+1,j}] \to \mathbb{R}^{n\times n}$, $j = \overline{1,k}$ such that (Natanson, 1960, p. 214)

$$\sum_{j=1}^{k} \int\limits_{t_{i,j}}^{t_{i+1,j}} \|F_j[\tau] - \widetilde{F}_{i,j}[\tau]\| \, \mathrm{d}\tau \leq \varepsilon_i, \qquad (23)$$

a Lipschitz continuous in the second argument matrix-function $\widetilde{G}_i[\cdot,\cdot]\colon [t_0,\vartheta] \times [t_i, t_{i+1}] \to \mathbb{R}^{n\times n}$ such that

$$\max_{\xi\in[t_0,\vartheta]} \int\limits_{t_i}^{t_{i+1}} \|G[\xi,\tau] - \widetilde{G}_i[\xi,\tau]\| \, \mathrm{d}\tau \leq \varepsilon_i, \qquad (24)$$

and the numbers

$$\alpha_i = \sum_{j=1}^{k} \Bigg(\int\limits_{t_{i,j}}^{t_{i+1,j}} \Big\| \frac{\mathrm{d}\widetilde{F}_{i,j}[\tau]}{\mathrm{d}\tau} \Big\| \, \mathrm{d}\tau + 2 \max_{\tau\in[t_{i,j}, t_{i+1,j}]} \|\widetilde{F}_{i,j}[\tau]\| \Bigg),$$

$$\beta_i = \max_{\xi\in[t_0,\vartheta]} \Bigg(\int\limits_{t_i}^{t_{i+1}} \Big\| \frac{\mathrm{d}\widetilde{G}_i[\xi,\tau]}{\mathrm{d}\tau} \Big\| \, \mathrm{d}\tau + 2 \max_{\tau\in[t_i,t_{i+1}]} \|\widetilde{G}_i[\xi,\tau]\| \Bigg),$$

$$\mu_i = (1 + \alpha_i + \beta_i) e^{\eta(\vartheta - t_0)}. \qquad (25)$$

Put

$$\nu = \frac{\varepsilon e^{-\eta(\vartheta - t_0)}}{3(\mu_{l-2} + \mu_{l-2}\mu_{l-3} + \ldots + \mu_{l-2}\mu_{l-3}\ldots\mu_0 + 1)}. \qquad (26)$$

Let a function $s[\cdot]$ satisfy (20). Denote

$$w_i = \sum_{j=1}^{k} \max_{t\in[t_0,t_{i+1}]} \Bigg\| \int\limits_{t_0}^{T[t-h_j]} F_j[\tau] \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \, \mathrm{d}\tau \Bigg\| + \max_{t\in[t_0,t_i]} \max_{\xi\in[t_0,\vartheta]} \Bigg\| \int\limits_{t_0}^{t} G[\xi,\tau] \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \, \mathrm{d}\tau \Bigg\|.$$

Due to (20), we have

$$\|s[t]\| \leq \nu + w_i + \eta \int\limits_{t_0}^{t} \|s[\tau]\| \, \mathrm{d}\tau, \quad t \in [t_0, t_{i+1}], \quad i = \overline{0, l-1}.$$

From this estimate, using Gronwall-Bellman lemma (Bellman and Cooke, 1963, p. 31), we deduce

$$\|s[t]\| \leq (\nu + w_i) e^{\eta(\vartheta - t_0)}, \; t \in [t_0, t_{i+1}], \; i = \overline{0, l-1}. \quad (27)$$

Note that, for any $i = \overline{0, l-2}$, the numbers w_i satisfy the following recursive estimate:

$$w_{i+1} \leq w_i + \sum_{j=1}^{k} \max_{t\in[t_{i+1},t_{i+2}]} \Bigg\| \int\limits_{t_{i,j}}^{T[t-h_j]} F_j[\tau] \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \, \mathrm{d}\tau \Bigg\| + \max_{t\in[t_i,t_{i+1}]} \max_{\xi\in[t_0,\vartheta]} \Bigg\| \int\limits_{t_i}^{t} G[\xi,\tau] \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \, \mathrm{d}\tau \Bigg\|. \qquad (28)$$

For the first integral, using (23) and (26), for any $t \in [t_{i+1}, t_{i+2}]$, we deduce

$$\sum_{j=1}^{k} \Bigg\| \int\limits_{t_{i,j}}^{T[t-h_j]} F_j[\tau] \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \, \mathrm{d}\tau \Bigg\| \qquad (29)$$

$$\leq \sum_{j=1}^{k} \int\limits_{t_{i,j}}^{t_{i+1,j}} \|F_j[\tau] - \widetilde{F}_{i,j}[\tau]\| \Big\| \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \Big\| \, \mathrm{d}\tau$$

$$+ \sum_{j=1}^{k} \|\widetilde{F}_{i,j}[T[t-h_i]]\| \|s[T[t-h_i]]\| + \sum_{j=1}^{k} \|\widetilde{F}_{i,j}[t_{i,j}]\| \|s[t_{i,j}]\|$$

$$+ \sum_{j=1}^{k} \int\limits_{t_{i,j}}^{t_{i+1,j}} \Big\| \frac{\mathrm{d}\widetilde{F}_{i,j}[\tau]}{\mathrm{d}\tau} \Big\| \|s[\tau]\| \mathrm{d}\tau \leq R_s \varepsilon_i + \alpha_i \max_{\tau\in[t_0,t_{i+1}]} \|s[\tau]\|.$$

Analogously, for the second integral, using (24) and (26), for any $t \in [t_i, t_{i+1}]$ and $\xi \in [t_0, \vartheta]$, we have

$$\Bigg\| \int\limits_{t_i}^{t} G[\xi,\tau] \frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau} \, \mathrm{d}\tau \Bigg\| \leq R_s \varepsilon_i + \beta_i \max_{\tau\in[t_0,t_{i+1}]} \|s[\tau]\|. \quad (30)$$

From (27)–(30), taking into account definition (26) of the numbers μ_i, we obtain

$$w_{i+1} \leq \mu_i (w_i + \nu) + 2R_s \varepsilon_i.$$

Applying this inequality for $i = l-2, l-3, \ldots, 0$, we deduce

$$w_{l-1} \leq 2R_s(\mu_{l-1}\varepsilon_{l-2} + \ldots + \mu_{l-1}\mu_{l-2}\mu_{l-3}\ldots\mu_1\varepsilon_0) + \big(\mu_{l-2} + \mu_{l-2}\mu_{l-3} + \ldots + \mu_{l-2}\mu_{l-3}\ldots\mu_0\big)\nu.$$

Then, from (22), (26) and (27), we derive (21). ■

6. AIMING PROCEDURE

Let us describe an aiming procedure between systems (1), (2) and (10), (11). The procedure is based on a partition of the control interval $[t_0, \vartheta]$:

$$\Delta_\delta = \big\{ t_j \colon 0 < t_{j+1} - t_j < \delta, \, j = \overline{0, J-1}, \, t_J = \vartheta \big\}. \; (31)$$

Realizations $u[\cdot]$ and $\hat{v}[\cdot]$ are formed according to the following feedback rule:

$$u[t] = u_j, \quad \hat{v}[t] = \hat{v}_j, \quad t \in [t_j, t_{j+1}), \quad j = \overline{0, J-1}, \quad (32)$$

where

$$u_j \in \operatorname*{argmin}_{u\in\mathbb{U}} \langle P[t_j]u, r[t_j]\rangle, \quad \hat{v}_j \in \operatorname*{argmax}_{v\in\mathbb{V}} \langle Q[t_j]v, r[t_j]\rangle,$$

$$r[t] = x[t] - y^{[0]}[t] - \int\limits_{t_0}^{t} \mathbf{L}_1(\tau, \dot{x}_\tau[\cdot] - z^{[0]}_\tau[\cdot]) \, \mathrm{d}\tau \qquad (33)$$

$$- \int\limits_{t_0}^{t} \mathbf{L}_2(\tau, x_\tau[\cdot] - y^{[0]}_\tau[\cdot]) \, \mathrm{d}\tau.$$

Theorem 2. For any number $\varepsilon > 0$, there exist numbers $\delta = \delta(\varepsilon) > 0$ and $M = M(\varepsilon) > 0$ such that, for any triple $(\theta, \varphi[\cdot], \psi[\cdot]) \in \Theta$, any natural number $m \geq M$ and any admissible realizations $\hat{u}[\cdot]$, $v[\cdot]$, if realizations $u[\cdot]$, $\hat{v}[\cdot]$ are formed according to aiming procedure (31)–(33), then, for the solution $x[\cdot]$ of systems (1), (2) and the solution $W[\cdot]$ of system (10), (11), the following inequality holds:

$$\|x[t] - y^{[0]}[t]\| \leq \varepsilon, \quad t \in [t_0, \vartheta].$$

Proof. Let us apply Lemma 2 for

$$s[t]=x[t]-y^{[0]}[t],\quad R_s=R_x+R_y,\quad \alpha_*=\max_{j=\overline{1,k}}\alpha_{1,j},$$

$$\mu=\sum_{j=0}^{k}\alpha_{2,j}+h\beta_2,\ \ F_j[t]=A_{1,j}[t+h_j],\ \ t\in[t_0,T[\vartheta-h_j]],$$

$$G[t,\tau]=\int\limits_{\max\{-h,\tau-t\}}^{0}B_1[\tau-\xi,\xi]\,\mathrm{d}\xi,\quad t,\tau\in[t_0,\vartheta],\quad (34)$$

where the numbers $\alpha_{2,j}$, β_2, R_x and R_y are taken from (4), (5), (8) and Lemma 1, respectively. Then inequality (19) and the first relation in (20) hold. Hence, to prove this theorem, we need to prove the second relation in (20).

By $r[t]$ from (33) let us defined function $V[t]=\|r[t]\|^2/2$, $t\in[t_0,\vartheta]$. Then we have

$$\|s[t]\|\le\Big\|\int\limits_{t_0}^{t}\mathbf{L}_1(\tau,\dot{x}_\tau[\cdot]-z^{[0]}_\tau[\cdot])\,\mathrm{d}\tau\Big\|$$

$$+\Big\|\int\limits_{t_0}^{t}\mathbf{L}_2(\tau,x_\tau[\cdot]-y^{[0]}_\tau[\cdot])\,\mathrm{d}\tau\Big\|+\sqrt{2V[t]}.\quad (35)$$

For the first term, using definition (3) of the operator $\mathbf{L}_1$, condition $(L.1)$ and relations (2), (10), (12), we deduce

$$\Big\|\int\limits_{t_0}^{t}\mathbf{L}_1(\tau,\dot{x}_\tau[\cdot]-z^{[0]}_\tau[\cdot])\,\mathrm{d}\tau\Big\|\quad (36)$$

$$\le\sum_{j=1}^{k}\Big\|\int\limits_{t_0}^{T[t-h_j]}A_{1,j}[\tau+h_j](\dot{x}[\tau]-\dot{y}^{[0]}[\tau])\,\mathrm{d}\tau\Big\|$$

$$+\Big\|\int\limits_{-h}^{-h_*}\int\limits_{t_0}^{T[t+\xi]}B_1[\tau-\xi,\xi](\dot{x}[\tau]-\dot{y}^{[0]}[\tau])\,\mathrm{d}\tau\,\mathrm{d}\xi\Big\|.$$

Then, changing the order of integration and using definitions (34) of $s[t]$, $F_j[t]$ and $G[t,\tau]$, we obtain

$$\Big\|\int\limits_{t_0}^{t}\mathbf{L}_1(\tau,\dot{x}_\tau[\cdot]-z^{[0]}_\tau[\cdot])\,\mathrm{d}\tau\Big\|\le\sum_{j=1}^{k}\Big\|\int\limits_{t_0}^{T[t-h_j]}F_j[\tau]\frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau}\,\mathrm{d}\tau\Big\|$$

$$+\Big\|\int\limits_{t_0}^{T[t-h_*]}G[t,\tau]\frac{\mathrm{d}s[\tau]}{\mathrm{d}\tau}\,\mathrm{d}\tau\Big\|.\quad (37)$$

For the second term in (35), by analogy with (36) and after that using inequalities (4), (5) and definition (34) of the number η, we deduce

$$\Big\|\int\limits_{t_0}^{t}\mathbf{L}_2(\tau,x_\tau[\cdot]-y^{[0]}_\tau[\cdot])\,\mathrm{d}\tau\Big\|\le\sum_{j=0}^{k}\Big\|\int\limits_{t_0}^{T[t-h_j]}A_{2,j}[\tau+h_j]s[\tau]\,\mathrm{d}\tau\Big\|$$

$$+\int\limits_{-h}^{0}\Big\|\int\limits_{t_0}^{T[t+\xi]}B_2[\tau-\xi,\xi]s[\tau]\,\mathrm{d}\tau\Big\|\mathrm{d}\xi\le\eta\int\limits_{t_0}^{t}\|s[\tau]\|\,\mathrm{d}\tau.\quad (38)$$

Let us estimate the third term in (35). The function $V[\cdot]$ is Lipschitz continuous. Taking into account equations (1), (10), for almost every $t\in[t_0,\vartheta]$, we have

$$\frac{\mathrm{d}V[t]}{\mathrm{d}t}=\langle\frac{\mathrm{d}r[t]}{\mathrm{d}t},r[t]\rangle=\langle\mathbf{L}_1(t,z^{[0]}_t[\cdot]-S(Z[t])[\cdot]),r[t]\rangle$$

$$+\langle\mathbf{L}_2(t,y^{[0]}_t[\cdot]-S(Y[t])[\cdot]),r[t]\rangle+\langle P[t]\big(u[t]-\hat{u}[t]\big),r[t]\rangle$$

$$+\langle Q[t](v[t]-\hat{v}[t]),r[t]\rangle.\quad (39)$$

Let us estimate $\langle P[t](u[t]-\hat{u}[t]),r[t]\rangle$. We deduce

$$\langle P[t](u[t]-\hat{u}[t]),r[t]\rangle\le\langle P[t_j](u[t]-\hat{u}[t]),r[t_j]\rangle$$

$$+(\|u[t]\|+\|\hat{u}[t]\|)\Big(\|P[t]-P[t_j]\|\|r[t]\|+\|P[t_j]\|\|r[t]-r[t_j]\|\Big).$$

From (4), (5), (20), (37) and (38) we have

$$r[\cdot]\in AC(R_r),\quad R_r=1+\sum_{j=1}^{k}\alpha_{1,j}+\beta_1+\sum_{j=0}^{k}\alpha_{2,j}+\beta_2.$$

Hence, taking into account continuity of the matrix-function $P[t]$, for sufficiently small $\delta>0$, for $t\in[t_j,t_{j+1})$ and $i=\overline{0,J-1}$, we obtain

$$\langle P[t](u[t]-\hat{u}[t]),r[t]\rangle\le\nu^2/(8(\vartheta-t_0))$$

$$+\langle P[t_j](u[t]-\hat{u}[t]),r[t_j]\rangle.\quad (40)$$

By (32), (33), we have $\langle P[t_j](u[t]-\hat{u}[t]),r[t_j]\rangle\le 0$ for $t\in[t_j,t_{j+1})$. Then, from (41) we deduce

$$\langle P[t](u[t]-\hat{u}[t]),r[t]\rangle\le\nu^2/(8(\vartheta-t_0)),\quad t\in[t_0,\vartheta].\ (41)$$

In the same way, we derive

$$\langle Q[t](v[t]-\hat{v}[t]),r[t]\rangle\le\nu^2/(8(\vartheta-t_0)),\quad t\in[t_0,\vartheta].\ (42)$$

From (39), (41) and (42), for almost every $t\in[t_0,\vartheta]$, we obtain

$$\frac{\mathrm{d}V[t]}{\mathrm{d}t}\le\nu^2/(4(\vartheta-t_0))+\langle\mathbf{L}_1(t,z^{[0]}_t[\cdot]-S(Z[t])[\cdot]),r[t]\rangle$$

$$+\langle\mathbf{L}_2(t,y^{[0]}_t[\cdot]-S(Y[t])[\cdot]),r[t]\rangle.\quad (43)$$

Due to (2) and (11) we have $V[t_0]=\|r[t_0]\|^2/2=0$. Then, according to (43) we obtain

$$V[t]\le\nu^2/4+\Big\|\int\limits_{t_0}^{t}\langle\mathbf{L}_1(\tau,z^{[0]}_\tau[\cdot]-S(Z[\tau])[\cdot]),r[\tau]\rangle\,\mathrm{d}\tau\Big\|$$

$$+\Big\|\int\limits_{t_0}^{t}\langle\mathbf{L}_2(t,y^{[0]}_\tau[\cdot]-S(Y[\tau])[\cdot]),r[\tau]\rangle\,\mathrm{d}\tau\Big\|.\quad (44)$$

By Theorem 1, there exists a number $M=M(\nu^2/8)>0$ such that, for any natural number $m\ge M$, we have $V[t]\le\nu^2/2$. Then, from (35)–(38) we derive (20). ∎

7. EXAMPLE

Consider a conflict-controlled dynamical system described by the linear neutral-type functional-differential equation

$$\begin{cases} \dot{x}_1[t] - 0.5\sin(t)\dot{x}_2[t-1] \\ \qquad = x_2[t] + \int\limits_{-h}^{0} \sin(t-\xi)x_1[t+\xi]\,\mathrm{d}\xi + v_1[t], \\ \dot{x}_2[t] - 0.5\cos(t)\dot{x}_1[t-1] \\ \qquad = -x_1[t] + \sin(t)x_2[t-0.5] + u[t] + v_2[t], \end{cases}$$

$$t \in [0,5], \quad x[t] = (x_1[t], x_2[t]) \in \mathbb{R}^2, \quad u[t] \in \mathbb{R}, \quad |u[t]| \le 1,$$
$$v[t] = (v_1[t], v_2[t]) \in \mathbb{R}^2, \quad \|v[t]\| \le 1.$$

with the initial condition

$$\begin{matrix} x_1[t_0] = 1, & x_1[\xi] = 0.5, & \dot{x}_1[\xi] = \cos(5\xi), \\ x_2[t_0] = 0, & x_2[\xi] = \cos(5\xi), & \dot{x}_2[\xi] = 1, \end{matrix} \quad \xi \in [-1,0].$$

For this system, according to (10), (11) we construct the approximating system, perform the aiming procedure (31)–(33) and simulate the situation with

$$\hat{u}[t] = \sin(3t), \quad v_1[t] = \sin(2t), \quad v_2[t] = \cos(2t), \quad t \in [0,5].$$

The results of the simulation are shown in Figures 1,2 and Table 1.

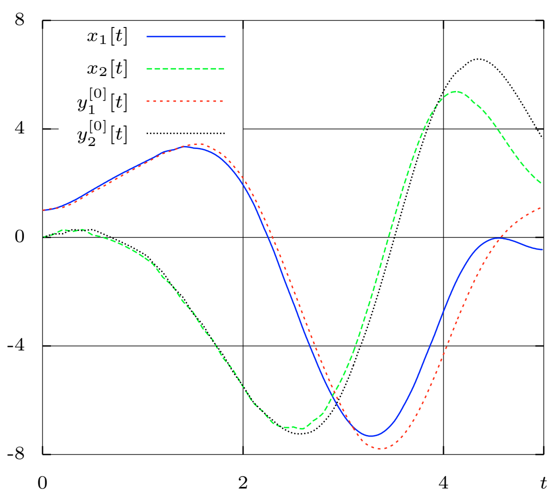

Fig. 1. Simulation results for $m = 10$, $\delta = 0.1$.

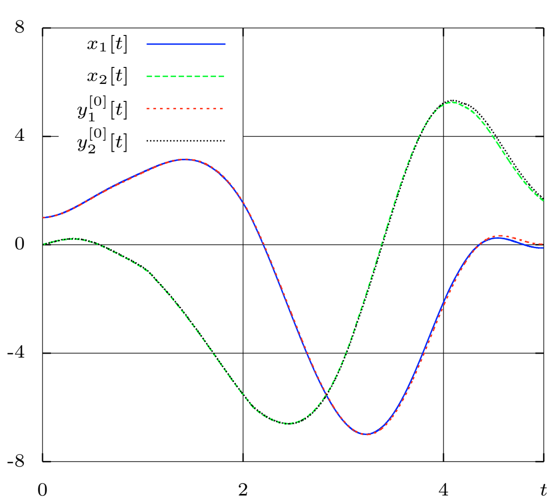

Fig. 2. Simulation results for $m = 100$, $\delta = 0.01$.

m	δ	$\max\limits_{t\in[t_0,\vartheta]} \|x[t] - y^{[0]}[t]\|$
10	0.1	2.422
50	0.02	0.474
100	0.01	0.196
200	0.005	0.104

Table 1. Simulation results.

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